On the Kähler-Ricci flow near a Kähler-Einstein metric

Song Sun and Yuanqi Wang

Abstract

Motivated by recent study of the convergence of Calabi flow near a constant scalar curvature Kähler metric, we prove a similar theorem on the stability of the Kähler-Ricci flow near a Kähler-Einstein metric of positive scalar curvature.

1 Introduction

This is a continuation of section 5 in [CS]. We shall prove the following

Theorem 1.1. *Let (M, ω, J, g) be a Kähler-Einstein manifold of Einstein constant 1, i.e. $Ric(g) = g$. Then there is a $C^{k,\gamma}(k \gg 1)$ tensor neighborhood $\mathcal{N}$ of (ω, J, g), such that the following holds. For any Kähler structure (ω', J', g') which lies in $\mathcal{N}$ and satisfies $\omega' \in 2\pi c_1(M; J')$, we denote by $(\omega(t), J(t), g(t))$ the (normalized) Kähler-Ricci flow starting from g'*

$$\begin{cases} \frac{\partial g(t)}{\partial t} = -Ric(g(t)) + g(t), & t \geq 0 \\ J(t) = J' & t \geq 0 \\ g(0) = g'. \end{cases} \tag{1}$$

Then there is an isotopy of diffeomorphisms f_t such that $f_t^*(\omega(t), J(t), g(t))$ converges in $C^{k,\gamma}$ to a Kähler-Einstein metric $(\omega_\infty, J_\infty, g_\infty)$ in a polynomial rate, i.e. there are constants $C > 0$, $\alpha > 0$, such that

$$||f_t^* g(t) - g_\infty||_{C^{k,\gamma}} + ||f_t^* J(t) - J_\infty||_{C^{k,\gamma}} \leq C \cdot (t+1)^{-\alpha}, \tag{2}$$

where the $C^{k,\gamma}$ topology is taken with respect to the initial Kähler-Einstein metric (ω, J, g). Moreover, if J is adjacent to J' in the sense that there is a sequence of diffeomorphisms ϕ_i such that $\phi_i^* J' \to J$(see [CS]), then the limit $(\omega_\infty, J_\infty, g_\infty)$ is isomorphic to (ω, J, g) under the action of the diffeomorphism group.

Remark 1.2. We should mention that two other approaches on the stability of Ricci flow on Fano manifolds have been previously announced by Arezzo-La Nave(see [AL]) and by Tian-Zhu(see [TZ3]). Our method here is different, and theorem 1.1 follows from a stability lemma for general modified Ricci flow (3.2).

R. Hamilton([Ha]) introduced the *Ricci flow* as a way to produce Einstein metrics. He proved the short time existence using his own generalization of the Nash-Moser inverse function theorem. Later De Turck reproved the short time existence by a gauge fixing trick. The long time existence for general Ricci flow fails. H-D. Cao([Cao]) first studied the Ricci flow on a Kähler manifold, called the *Kähler-Ricci flow*. The Kähler condition is preserved under the flow, and the equation can be reduced to a scalar equation on the Kähler potential. Using Yau's estimates for complex Monge-Ampère equations, Cao proved the long time existence of the flow in any Kähler class proportional to the canonical class, and proved convergence in the case of non-positive first Chern class. In the case of positive first Chern class, the Kähler-Ricci flow in the anti-canonical class $2\pi c_1(M)$ takes the normalized form as in (1) with the complex structure fixed. In this case the flow does not necessarily converge to a Kähler-Einstein metric, due to the obstructions to the existence of Kähler-Einstein metrics on Fano manifolds. In an unpublished paper, Perelman proved that the scalar curvature and diameter are always uniformly bounded along the Kähler-Ricci flow, and he announced that if there exists a Kähler-Einstein metric, then the flow starting from any Kähler metric in the anti-canonical class of the same complex structure converges to a Kähler-Einstein metric(Note this was previously proved by Chen-Tian([CT]) when the initial metric is assumed to have positive bisectional curvature). In [TZ1], using Perelman's estimates, an alternative proof of this convergence was given. The case we consider is a stability type theorem: the Kähler-Ricci flow initiating from a Kähler metric near a Kähler-Einstein metric(possibly with a different complex structure) always converges(modulo diffeomorphisms) polynomially fast to a Kähler-Einstein metric. Using the gradient interpretation of Ricci flow by Perelman([Pe]), Tian-Zhu([TZ2]) previously proved such a stability theorem when the Kähler-Einstein metric is assumed to be isolated. Our theorem is based on a more close investigation of this gradient nature, and is motivated by the case of Calabi flow studied in [CS]. We believe similar idea may be helpful in the study of other geometric flows. Note the gauging diffeomorphisms must diverge, if there is no Kähler-Einstein metric compatible with the complex structure J'. There are also various studies on the stability of the Ricci flow near a Ricci flat metric, for instance Sesum([Se1]) proved exponential convergence of the Ricci flow near a Ricci flat metric which is linearly stable and integrable. For more details on this topic, readers are referred to [Se1] and the references therein. In our case, the convergence is exponential precisely when the complex structure J' itself admits a Kähler-Einstein metric. In general, there are obstructions to deform Kähler-Einstein metrics, so the exponential convergence fails.

The idea of the proof is as follows. Perelman discovered that the Ricci flow is essentially a gradient flow. Let $\mathcal{R}(M)$ be the space of all Riemannian metrics on M. There is an appropriate Riemannian metric defined on $\mathcal{R}(M)$,

which is invariant under the action of the diffeomorphism group $\mathrm{Diff}(M)$. The Ricci flow direction differs from the gradient of the so-called μ functional only by an infinitesimal action of $\mathrm{Diff}(M)$. Both the Ricci flow and the μ functional are $\mathrm{Diff}(M)$-invariant, and thus live on $\mathcal{R}(M)/\mathrm{Diff}(M)$. Then on this quotient, the Ricci flow in the normalized form as in (1) is exactly the gradient flow of the μ functional. If $[g_0]$ is a local maximum of μ on $\mathcal{R}(M)/\mathrm{Diff}(M)$, then we ask whether the Ricci flow initiating from a nearby point would always stay close to $[g_0]$ and converge to a maximum point of μ at infinity. If we were in finite dimension, then by Lojasiewicz's fundamental structure theorem for real analytic functions, the flow would converge polynomially fast to a unique limit at infinity provided the functional is real-analytic. The limit is also a local maximum, but possibly be different from the one we start with if there is a non-trivial moduli. Note in the case when the maximum is non-degenerate in the sense of Morse-Bott, then indeed we have exponential convergence. But if the maximum is degenerate, then we can not expect exponential convergence, and the rate of convergence depends on how bad the degeneracy is. For more details about the finite dimensional setting, the readers are referred to [CS]. Our problem has an infinite dimensional nature, but as in [Si], the real difficulty is still finite dimensional. Heuristically, one can decompose the tangent space at $[g_0]$ into the direct sum of two parts. One part is infinite dimensional but on which the Hessian of μ is non-degenerate, and the other part is finite dimensional on which we can apply Lojasiewicz's inequality. We need to check the functional μ is real-analytic near a Kähler-Einstein metric. The last problem is that our Kähler-Einstein metric may not be a local maximum of μ among all variations, but is so among all Kähler metrics in the same real cohomology class. This is enough for our purpose, thanks to the fact that such a subset is preserved under the Ricci flow.

This note is organized as follows. In section 2, we set up some notations and definitions. In section 3, we prove a Lojasiewicz type inequality for the μ functional(lemma 3.1), and prove a general stability theorem for the modified Ricci flow(lemma 3.2). In section 4, we prove theorem 1.1, using results in section 3.

Acknowledgements: Both authors would like to thank Professor Xiuxiong Chen for constant support. We also thank Prof. C. Arezzo and Prof. X-H. Zhu for sending us their results(see [AL], [TZ3]). The first author would also like to thank the department of Mathematics in Stony Brook for its hospitality during the year 2009-2010. The first author is partially supported by an Research Assistantship in an NSF grant.

2 Preliminaries

Suppose M is an n-dimensional smooth compact manifold. Denote by $\mathcal{R}(M)$ the space of all Riemannian metrics on M. This is an open convex subset of the linear space of all smooth sections of symmetric 2-tensors on M. Later we will assign either the $C^{k,\gamma}$ topology or L^2_k topology on $\mathcal{R}(M)$. Right now we simply take the C^∞ topology. For any smooth measure dm on M, we denote by $C_0^\infty(M; dm; \mathbb{R})(C_0^{k,\gamma}(M; dm; \mathbb{R}))$ the space of $C^\infty(C^{k,\gamma})$ real-valued functions f on M satisfying

$$\int_M e^{-f} dm = 1.$$

Given a Riemannian metric g, We define Perelman's functional

$$\mathcal{W}_g(f) = \int_M [\frac{1}{2}(|\nabla f|^2 + R(g)) + f]e^{-f} \,\mathrm{dVol}_g, \tag{3}$$

for any $f \in C_0^\infty(M; dm; \mathbb{R})$. We will denote this functional either by $\mathcal{W}_g(f)$ or $\mathcal{W}(g, f)$, where in the latter case we emphasize $\mathcal{W}$ as a function depending on both the metric g and the function f. Then by a straightforward calculation the first variation of $\mathcal{W}$ is given by

Lemma 2.1.

$$\begin{aligned} \delta\mathcal{W}(h, v) = \quad & \int_M [-\frac{1}{2} < Ric(g) + \mathrm{Hess}\, f - (\Delta f - \frac{1}{2}|\nabla f|^2 + f + \frac{1}{2}R(g))g, h >_g \\ & -(\Delta f - \frac{1}{2}|\nabla f|^2 + f + \frac{1}{2}R(g) - 1)v]e^{-f} \,\mathrm{dVol}_g \,. \end{aligned} \tag{4}$$

For a given Riemannian metric g, we define Perelman's μ functional to be

$$\mu(g) = \inf_{f \in C_0^\infty(M; \mathrm{dVol}_g; \mathbb{R})} \mathcal{W}_g(f) \tag{5}$$

By a minimizing procedure(see [Ra]) the infimum is always achievable by a function f which possesses the same regularity as the metric g. From equation (4) we see that f satisfies the non-linear equation

$$\Delta f - \frac{1}{2}|\nabla f|^2 + f + \frac{1}{2}R(g) = \mu(g). \tag{6}$$

We call a metric $g \in \mathcal{R}(M)$ *regular* if there is a neighborhood $\mathcal{U}$ of g, such that for any $g' \in \mathcal{U}$, the minimizer of $\mathcal{W}_{g'}$ is unique and depends real-analytically on g'. The following lemma will be proved in the next section.

Lemma 2.2. *A normalized shrinking gradient Ricci soliton(i.e. $Ric(g) + \mathrm{Hess}\, f = g$) is regular.*

Now we assume g is regular, then it is easy to see from (4) the first variation of μ is given by

$$\delta\mu(h) = -\frac{1}{2}\int_M < Ric(g) + \text{Hess}\, f - g, h >_g e^{-f}\, \text{dVol}_g, \tag{7}$$

where f is the minimizer of $\mathcal{W}_g$. If we endow $\mathcal{R}(M)$ with the Riemannian metric

$$(h_1, h_2)_g := \frac{1}{2}\int_M < h_1, h_2 >_g e^{-f}\, \text{dVol}_g, \tag{8}$$

then we see that

$$\nabla\mu(g) = -(Ric(g) + \text{Hess}\, f - g) \tag{9}$$

So the critical points of μ are exactly normalized shrinking gradient Ricci solitons. The gradient flow of μ functional is

$$\begin{cases} \frac{\partial g(t)}{\partial t} = -Ric(g(t)) + g(t) - \text{Hess}_t\, f(t), \quad t \geq 0 \\ g(0) = g. \end{cases} \tag{10}$$

We shall call this flow the *"modified" Ricci flow.* We have

Lemma 2.3. *Up to an isotopy of diffeomorphisms, the gradient flow (10) is equivalent to the normalized Ricci flow*

$$\begin{cases} \frac{\partial g(t)}{\partial t} = -Ric(g(t)) + g(t), \quad t \geq 0 \\ g(0) = g, \end{cases} \tag{11}$$

if we assume $g(t)$ is regular for all time as long as the flow exists.

Proof. This is because under our hypothesis the function $f(t)$ depends smoothly on t, and then we can translate between the two flows by the isotopy of diffeomorphisms generated by $\nabla_t f(t)$. $\square$

3 A stability lemma for the Ricci flow

We shall prove in the end of this section the following Lojasiewicz type inequality.

Lemma 3.1. *Let g_0 be a normalized shrinking gradient Ricci soliton. Then there is a $C^{k,\gamma}(k \gg 1)$ neighborhood $\mathcal{U}$ of g_0 in $\mathcal{R}(M)$, and constant $C > 0$, and $\alpha \in [\frac{1}{2}, 1)$, such that for any $g \in \mathcal{U}$, we have*

$$||\nabla\mu(g)|| \geq C \cdot |\mu(g_0) - \mu(g)|^{\alpha}, \tag{12}$$

For convenience, we denote

$$\mathcal{V}_\delta^{k,\gamma} = \{g \in \mathcal{R}(M) | ||g - g_0||_{C^{k,\gamma}} \leq \delta\}.$$

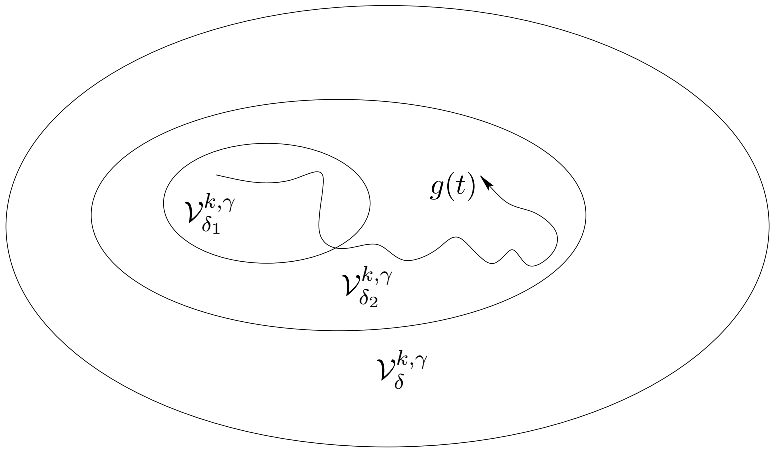

Figure 1: The flow $g(t)$ can never exit $\mathcal{V}_{\delta_2}^{k,\gamma}$

Lemma 3.2. *Suppose g_0 is a normalized shrinking gradient Ricci soliton. Then there exist $\delta_2 > \delta_1 > 0$, such that for any $g \in \mathcal{V}_{\delta_1}^{k,\gamma}$, the modified Ricci flow*

$$\begin{cases} \frac{\partial g(t)}{\partial t} = -Ric(g(t)) + g(t) - \mathrm{Hess}_t f(t), & t \geq 0 \\ g(0) = g, & t = 0 \end{cases} \tag{13}$$

starting from any g satisfies $g(t) \in \mathcal{V}_{\delta_2}^{k,\gamma}$ as long as $\mu(g(t)) \leq \mu(g_0)$. In particular, if we know a priori that $\mu(g(t)) \leq \mu(g_0)$ for all t, then the flow $g(t)$ exists globally for all time t, and converges in $C^{k,\gamma}$ to a limit g_∞ which is also a shrinking gradient Ricci soliton with $\mu(g_\infty) = \mu(g_0)$. The convergence is in a polynomial rate.

Proof. The proof is by Lojasiewicz arguements(see [CS]). For convenience, we include the details here. Choose k large and $\delta > 0$ small such that all metrics in $\mathcal{V}_{\delta}^{k,\gamma}$ are regular with (12) true, and the short time existence is uniform up to time 1 for the flow starting from metrics in $\mathcal{V}_{\delta}^{k,\gamma}$. It suffices to prove that there exists $0 < \delta_1 < \delta_2 < \delta$ such that for any modified Ricci flow solution $g(t)$ with $g(0) \in \mathcal{V}_{\delta_1}^{k,\gamma}$, if $g(t) \in \mathcal{V}_{\delta}^{k,\gamma}$ and $\mu(g(t)) \leq \mu(g_0) = 0$ for $t \in [0,T)(T > 1)$, then $g(T) \in \mathcal{V}_{\delta_2}^{k,\gamma}$(see figure 1). By the fundamental curvature estimates of Hamilton, for any integer $l \geq 1$, and $t \geq 1$, we have

$$||Rm(g(t))||_{C_t^{l,\gamma}} \leq C(l),$$

where the norm is with respect to the metric $g(t)$. Since all metrics in $\mathcal{V}_{\delta}^{k,\gamma}$ are regular, this gives

$$||f(t)||_{C_t^{l,\gamma}} \leq C(l). \tag{14}$$

Fix $\beta \in (2-\frac{1}{\alpha}, 1)$, then by interpolation inequalities for tensors, for any integer $p \geq 1$ there is an $N(p)$(independent of $t \geq 1$), such that

$$\begin{aligned} ||\dot{g}(t)||_{L^2_p(t)} &\leq C(p) \cdot ||\dot{g}(t)||^{\beta}_{L^2(t)} \cdot ||Ric(g(t)) - g(t) + \mathrm{Hess}_t f(t)||^{1-\beta}_{L^2_{N(p)}(t)} \\ &\leq C(p) \cdot ||\dot{g}(t)||^{\beta}_{L^2(t)}. \end{aligned}$$

Since

$$\begin{aligned} & \frac{d}{dt}[\mu(g_0) - \mu(g(t))]^{1-(2-\beta)\alpha} \\ = & -[1-(2-\beta)\alpha] \cdot [\mu(g_0) - \mu(g(t))]^{-(2-\beta)\alpha} \cdot ||\nabla \mu(g(t))||^2 \\ \leq & -C \cdot ||\nabla \mu(g(t))||^{\beta}_{L^2(t)}, \end{aligned} \tag{15}$$

we get

$$\begin{aligned} \int_1^T ||\dot{g}(t)||^{\beta}_{L^2(t)} dt &\leq C(\beta) \cdot [\mu(g_0) - \mu(g(1))]^{1-(2-\beta)\alpha} \\ &\leq C(\beta) \cdot [\mu(g_0) - \mu(g(0))]^{1-(2-\beta)\alpha}. \end{aligned} \tag{16}$$

Therefore

$$\int_1^T ||\dot{g}(t)||_{L^2_p(t)} dt = C(p) \cdot \epsilon(\delta_1),$$

where $\epsilon(\delta_1) \to 0$ as $\delta_1 \to 0$. Since the Sobolev constant is uniformly bounded in $\mathcal{V}^{k,\gamma}_{\delta}$, we obtain for any $l \geq 1$,

$$\int_1^T ||\dot{g}(t)||_{C^{l,\gamma}_t} dt = C(l) \cdot \epsilon(\delta_1), \tag{17}$$

and

$$||g(T) - g(1)||_{C^{k,\gamma}_t} \leq \int_1^T ||\dot{g}(t)||_{C^{k,\gamma}_t} dt = \epsilon(\delta_1).$$

Since the $C^{k,\gamma}$ norm defined by metrics in $\mathcal{V}^{k,\gamma}_{\delta}$ are equivalent to each other, we obtain

$$||g(T) - g(1)||_{C^{k,\gamma}} = \epsilon(\delta_1).$$

By the finite time stability of Ricci flow, we have

$$||g(1) - g_0||_{C^{k,\gamma}} = \epsilon(\delta_1).$$

Therefore,

$$||g(T) - g_0||_{C^{k,\gamma}} = \epsilon(\delta_1).$$

Now choose $\delta_2 = \frac{\delta}{2}$, and $\epsilon(\delta_1) \leq \delta_2$, then the first part of the lemma is proved.

Now we assume $\mu(g(t)) \leq \mu(g_0)$ for all t, then $g(t)$ exists for all time. Indeed, $g(t)$ can never exit $\mathcal{N}_{\delta_2}^{k,\gamma}$. Since

$$\frac{d}{dt}[\mu(g_0) - \mu(g(t))]^{1-2\alpha} \geq C > 0,$$

we have a decay estimate

$$\mu(g_0) - \mu(g(t)) \leq C \cdot (t+1)^{-\frac{1}{2\alpha-1}}. \tag{18}$$

Then for any $t_2 \geq t_1$, the same argument as before shows

$$\begin{aligned} ||g(t_1) - g(t_2)||_{C^{k,\gamma}} &\leq \int_{t_1}^{t_2} ||\dot{g}(t)||_{C^{k,\gamma}} dt \\ &\leq C(\beta) \cdot [\mu(g_0) - \mu(g_{t_1})]^{1-(2-\beta)\alpha} \\ &\leq C(\beta) \cdot (t_1+1)^{-\frac{1-(2-\beta)\alpha}{2\alpha-1}}. \end{aligned}$$

Hence the flow $g(t)$ converges uniformly in $C^{k,\gamma}$ to a limit g_∞, and let $t_2 \to \infty$ we obtain polynomial decay rate. By (16) and (18), we see that $\nabla\mu(g_\infty) = 0$, and $\mu(g_\infty) = \mu(g_0)$. So g_∞ is a shrinking gradient Ricci soliton. $\square$

Now we prove lemma 2.2. We first show the minimizer of $\mathcal{W}_g$ is unique if g is a shrinking gradient Ricci soliton.

Lemma 3.3. *If g is a shrinking gradient Ricci soliton:*

$$Ric(g) + \operatorname{Hess} f = g$$

with $\int_M e^{-f}\, \mathrm{dVol}_g = 1$, then the minimizer of $\mathcal{W}_g$ is unique and is equal to f. In particular, if $Ric(g) = g$, then the minimizer of $\mathcal{W}_g$ is $\log \mathrm{Vol}_g(M)$.

Proof. Let ϕ_t be the one-parameter group of diffeomorphisms generated by ∇f, and denote $g(t) = \phi_t^* g$. Then

$$\frac{\partial}{\partial t} g(t) = \operatorname{Hess}_t \phi_t^* f = g(t) - Ric(g(t)).$$

Suppose v is any minimizer of $\mathcal{W}_g$, Now solve the backward heat equation

$$\begin{cases} \frac{\partial v(t)}{\partial t} = \frac{1}{2}[n - R(g(t)) - \Delta_t v(t)] + |\nabla_t v(t)|^2, \quad t \in [0,\tau] \\ v(\tau) = \phi_\tau^* v \end{cases} \tag{19}$$

in a small time interval $[0,\tau]$. Let ψ_t be the isotopy of diffeomorphisms generated by the (time-dependent) vector fields $-\nabla_t v(t)$. Denote $\tilde{g}(t) = \psi_t^* g(t)$, and $\tilde{v}(t) = \psi_t^* v(t)$, then

$$\frac{\partial \tilde{g}(t)}{\partial t} = \tilde{g}(t) - Ric(\tilde{g}(t)) - \operatorname{Hess}_t \tilde{v}(t),$$

and

$$\frac{d\tilde{v}}{dt} = \frac{1}{2}[n - R(\tilde{g}(t)) - \Delta_t \tilde{v}(t)].$$

Then by (4) we have

$$\frac{\partial}{\partial t}\mathcal{W}(\tilde{g}(t), \tilde{v}(t)) = \frac{1}{2}\int_M |\tilde{g}(t) - Ric(\tilde{g}(t)) - \mathrm{Hess}_t\, \tilde{v}(t))|^2 e^{-\tilde{v}(t)}\, \mathrm{dVol}_t \geq 0.$$

However, since all $\tilde{g}(t)$ are in the same $\mathrm{Diff}(M)$ orbit, we have $\mu(\tilde{g}(t)) \equiv \mu(g)$. Since $\phi_\tau^* g$ is a minimizer of $\mathcal{W}_{g(\tau)}$, we have

$$\mathcal{W}(\tilde{g}(t), \tilde{v}(t)) \equiv \mathcal{W}(g(\tau), \phi_\tau^* v) = \mathcal{W}(g, v).$$

Hence

$$g - Ric(g) - \mathrm{Hess}\, v = 0,$$

and so $v = f$. □

In general, for any function f, we denote by d^{*f} and ∇^{*f} the adjoint of the d and ∇ with respect to the measure $e^{-f}\,\mathrm{dVol}_g$. Perelman([Pe]) observed the following Bochner formula relating the twisted Hodge Laplacian $\Delta_H^f = d^{*f}d + dd^{*f}$ and rough Laplacian $\Delta^f = -\nabla^{*f}\nabla$:

$$-\Delta^f \xi = \Delta_H^f \xi - Ric^f(\xi), \tag{20}$$

where ξ is any one-form and $Ric^f = Ric + \mathrm{Hess}\, f$. This gives immediately the following

Lemma 3.4. *Suppose $Ric(g_0) + \mathrm{Hess}\, f_0 = g_0$, then the first nonzero eigenvalue of $-\Delta^{f_0}$ acting on functions is greater than one.*

Proof of lemma 2.2. Suppose $Ric(g_0) + \mathrm{Hess}\, f_0 = g_0$. Define

$$L : C^{k,\gamma}(\mathcal{R}(M)) \times C^{k,\gamma}(M;\mathbb{R}) \to C^{k-4,\gamma}(M;\mathbb{R}),$$

which sends (g, f) to

$$\Delta_g^f(\Delta_g f + f - \frac{1}{2}|\nabla_g f|^2 + \frac{1}{2}R(g)) + \int_M e^{-f}\,\mathrm{dVol}_g - 1.$$

Then L is a real-analytic map between Banach manifolds. We have $L(g_0, f_0) = 0$, and the differential of L at (g_0, f_0) has its second component equal to

$$DL_{(g_0,0)}(0, f) = \Delta_{g_0}^{f_0}(\Delta_{g_0}^{f_0} f + f) - \int_M f\,\mathrm{dVol}_{g_0}. \tag{21}$$

This is an isomorphism by lemma 3.4. Thus by the real-analytic version of the implicit function theorem for Banach manifolds, there is a $C^{k,\gamma}$ neighborhood $\mathcal{V}_1$ of g_0 in $\mathcal{R}(M)$ and a real-analytic map $P : \mathcal{V}_1 \to C^{k,\gamma}(M;\mathbb{R})$ such that for any $g \in \mathcal{V}_1$, $P(g) \in C_0^{k,\gamma}(M; \mathrm{dVol}_g; \mathbb{R})$, and

$$L(g, P(g)) = 0.$$

Moreover, there is a $\delta > 0$ such that if $L(g, f) = 0$ for some $g \in \mathcal{V}_1$ and $||f||_{C^{k,\gamma}} \leq \delta$, then $f = P(g)$. Now $P(g) \in C_0^{k,\gamma}(M; \mathrm{dVol}_g; \mathbb{R})$ satisfies the equation

$$\Delta_g P(g) + P(g) - \frac{1}{2}|\nabla_g P(g)|^2 + \frac{1}{2}R(g) = \text{constant}.$$

In particular, $P(g)$ is a critical point of $\mathcal{W}_g$. Now we claim that we can choose $\mathcal{V}_2 \subset \mathcal{V}_1$, such that for any $g \in \mathcal{V}_2$, the minimizer for $\mathcal{W}_g$ is unique and is equal to $P(g)$. Suppose this were false, then there would be a sequence $g_i \in \mathcal{V}_1$, and a minimizer f_i of W_{g_i}, such that $g_i \to g_0$ in $C^{k,\gamma}$ topology, and $f_i \neq P(g_i)$ for any i. Let $u_i = e^{-\frac{f_i}{2}}$, then u_i is a minimizer of the functional

$$\widetilde{\mathcal{W}}_{g_i}(u) = \frac{1}{2}\int_M 4|\nabla_i u|^2 + R(g_i)u^2 - 2u^2 \log u^2 \, \mathrm{dVol}_{g_i}$$

under the constraint

$$\int_M u_i^2 \, \mathrm{dVol}_{g_i} = 1.$$

Moreover, we have $\widetilde{\mathcal{W}}_{g_i}(u_i) = \mu(g_i)$ and

$$-4\Delta_i u_i + R(g_i)u_i - 4u_i \log u_i - 2\mu(g_i)u_i = 0.$$

By definition, Perelman's μ functional is upper semi-continuous, i.e.

$$\overline{\lim_{i\to\infty}} \mu(g_i) \leq \mu(g_0).$$

Thus for all i we have

$$\int_M 4|\nabla_i u_i|^2 + R(g_i)u_i^2 - 4u_i^2 \log u_i \, \mathrm{dVol}_{g_i} = 2\mu(g_i) \leq C.$$

Here C is a constant which does not depend on i, but might vary from line to line. Choose $\epsilon > 0$ small such that $2 + 2\epsilon = \frac{2n}{n-2}$. By Jensen's inequality,

$$\int_M 2u_i^2 \log u_i \, \mathrm{dVol}_{g_i} = \frac{1}{\epsilon}\int_M u_i^2 \log u_i^{2\epsilon} \, \mathrm{dVol}_{g_i} \leq \frac{1}{\epsilon} \log \int_M u_i^{2+2\epsilon} \, \mathrm{dVol}_{g_i}.$$

Since $g_i \in \mathcal{V}_1$, we have a uniform bound of the Sobolev constant, so

$$\int_M u_i^{2+2\epsilon} \, \mathrm{dVol}_{g_i} \leq C \cdot (||\nabla_i u_i||_{L^2} + ||u_i||_{L^2})^{2+2\epsilon}.$$

From these we obtain that

$$||\nabla_i u_i||_{L^2} \leq C,$$

and thus

$$||u_i \log u_i||_{L^2} \leq C.$$

Then

$$|\mu(g_i)| \le C.$$

As in [Ra], by elliptic regularity, we obtain

$$||u_i||_{L^\infty} \le C,$$

and then for any $\beta \in (0,1)$,

$$||u_i||_{C^{1,\beta}} \le C.$$

Note by assumption the $C^{k,\gamma}$ norm defined by all the metrics g_i are equivalent to that defined by the metric g_0. By passing to a subsequence we can assume u_i converges to a limit u_∞ in $C^{1,\beta}$. Since u_i is a positive minimizer of $\widetilde{\mathcal{W}}_{g_i}$, u_∞ is a non-negative minimizer of $\widetilde{\mathcal{W}}_{g_0}$. By the strong maximum principle in [Ra], u_∞ is strictly positive. So we obtain a uniform positive lower bound for u_i. Then we get a uniform $C^{1,\beta}$ bound on f_i. Now by applying elliptic regularity for f_i, we get

$$||f_i||_{C^{k,\gamma}} \le C.$$

So by passing to a subsequence, f_i converges in weak $C^{k,\gamma}$ topology to f_∞. Then it is easy to see that $f_\infty = -2\log u_\infty$ is a minimizer for $\mathcal{W}_{g_0}$. By lemma 3.3 we obtain $f_\infty = f_0$. Note again implicit function theorem ensures that for i sufficiently large any critical point of $\mathcal{W}_{g_i}$ which is $C^{k,\gamma'}$(γ' slightly smaller than γ) close to 0 must be $P(g_i)$. Thus $f_i = P(g_i)$, and we arrive at a contradiction. $\square$

Proof of lemma 3.1 Denote by $\mathcal{R}'(M)$ the space of regular Riemannian metrics on M. $\mathcal{R}'(M)$ is endowed with the Riemannian metric given by equation (8). We put the L^2_k topology on $\mathcal{R}'(M)$(for convenience we do not use the Hölder norm here) and denote the completion by $\mathcal{R}'_k(M)$, so that $\mathcal{R}'_k(M)$ becomes a Banach manifold. Near a point g, $\mathcal{R}'_k(M)$ can be identified with an open set in the Banach space $\Gamma_k(g)$, which is the space of L^2_k sections of the bundle of symmetric two-tensors h. This gives rise to a coordinate chart for $\mathcal{R}'_k(M)$. We shall then identify any $h \in \mathcal{R}'_k(M)$ close to g with its "coordinate" $h \in \Gamma_k(g)$, by abuse of notation.

Let $\mathrm{Diff}_{k+1}(M)$ be the group of L^2_{k+1} diffeomorphisms of M. It acts on $\mathcal{R}'(M)$, preserving the Riemannian metric. Notice that both μ and $||\nabla\mu||$ are invariant under the action of $\mathrm{Diff}_{k+1}(M)$. In a neighborhood of the identity map, $\mathrm{Diff}_{k+1}(M)$ is modelled on the linear space $\mathfrak{g}_{k+1}$ of L^2_{k+1} vector fields X on M. At g_0, the tangent space to the $\mathrm{Diff}_{k+1}(M)$ orbit of g_0 is given by the image of

$$\mathcal{L}_k : \mathfrak{g}_{k+1} \to \Gamma_k(g_0); X \mapsto \nabla^s X,$$

where $\nabla^s X$ is the symmetrization of ∇X. The normal space to $\mathrm{Im}\,\mathcal{L}_k$ with respect to the Riemannian metric is given by $\mathrm{Ker}\,\mathcal{L}^*_k = \ker \nabla^{*f_0}$, which

consists of L^2_k divergence free symmetric 2-tensors. By the standard slice theorem(see for example [Eb]), there are neighborhoods $\mathcal{U} \subset \mathcal{V}$ of g_0 in $\mathcal{R}_k(M)$ for any $g \in \mathcal{U}$, there is a $\phi \in \mathrm{Diff}_{k+1}(M, dm)$ close to identity such that $\phi^* g = h \in \mathcal{V} \cap \mathrm{Ker}\,\mathcal{L}^*_k$. Here we implicitly made use of the identification mentioned above. Thus it suffices to prove inequality (12) for g in $\mathcal{Q} = \mathcal{V} \cap \mathrm{Ker}\,\mathcal{L}^*_k \subset \Gamma_k(g_0)$. We still denote by μ its restriction to $\mathcal{Q}$, and $\nabla_{\mathcal{Q}}\mu$ the gradient of μ on $\mathcal{Q}$, with respect to the constant metric on $\mathcal{Q}$ induced from $\Gamma_k(g_0)$. Then $\nabla_{\mathcal{Q}}\mu \in \mathrm{Ker}\,\mathcal{L}^*_{k-2}$ and we have for any $g \in \mathcal{Q}$

$$||\nabla\mu(g)|| \geq C \cdot ||\nabla_{\mathcal{Q}}\mu(g)||_{L^2}.$$

So it suffices to prove

Lemma 3.5. *There exists a neighborhood $\mathcal{N} \subset \mathcal{Q}$ of g_0 and constant $C > 0$, $\alpha > 0$, such that for any $g \in \mathcal{N}$, we have*

$$||\nabla_{\mathcal{Q}}\mu(g)|| \geq C \cdot |\mu(g) - \mu(g_0)|^{\alpha}. \tag{22}$$

Proof. We follow the same pattern of arguments as in [CS]. By (9), 0 is a critical point of μ. By a direct computation, the Hessian of μ at 0 (viewed as an operator from $\mathrm{Ker}\,\mathcal{L}^*_k$ to $\mathrm{Ker}\,\mathcal{L}^*_{k-2}$) is given by:

$$H_0(h) \quad = \quad \frac{1}{2}\Delta^{f_0} h + Rm(g_0) \circ h, \tag{23}$$

which is a “twisted ” Lichnerowicz Laplacian. Then H_0 is an elliptic differential operator of order 2 on $\mathrm{Ker}\,\mathcal{L}^*_k$. So it has a finite dimensional kernel W_0 which consists of smooth elements, and we have the following decomposition:

$$\mathrm{Ker}\,\mathcal{L}^*_k = W_0 \oplus W'_k,$$

where H_0 restricts to an invertible operator from W'_k to W'_{k-2}. So there exists a $c > 0$, such that for any $\eta' \in W'$, we have

$$||H_0(\eta')||_{L^2_{k-2}} \geq C \cdot ||\eta'||_{L^2_k}.$$

By the implicit function theorem, there are small constants $\epsilon_1, \epsilon_2 > 0$, such that for any $h_0 \in W_0$ with $||h_0||_{L^2} \leq \epsilon_1$(so $||h_0||_{L^2_{k-2}}$ is also small since W_0 is finite dimensional), there exists a unique element $\eta' = G(\eta_0) \in W'$ with $||\eta'||_{L^2_k} \leq \epsilon_2$, such that $\nabla_{\mathcal{Q}}\mu(\eta_0 + \eta') \in W_0$. Moreover by construction the map $G : B_{\epsilon_1}W_0 \to B_{\epsilon_2}W'$ is real analytic. Now consider the function

$$F : W_0 \to \mathbb{R}; \eta_0 \mapsto \mu(\eta_0 + G(\eta_0)).$$

This is a real analytic function with $\eta_0 = 0$ as a critical point. For any $\eta_0 \in W_0$, it is easy to see that $\nabla F(\eta_0) = \nabla_{\mathcal{Q}}\mu(\eta_0 + G(\eta_0)) \in W_0$.

Now we shall estimate the two sides of the inequality (22) separately. For any $\eta \in W$ with $||\eta||_{L^2_k} \leq \epsilon$, we can write $\eta = \eta_0 + G(\eta_0) + \eta'$, where $\eta_0 \in W_0$, $\eta' \in W'$, and

$$||\eta_0||_{L^2_k} \leq c \cdot ||\eta||_{L^2_k},$$
$$||G(\eta_0)||_{L^2_k} \leq c \cdot ||\eta||_{L^2_k},$$
$$||\eta'||_{L^2_k} \leq c \cdot ||\eta||_{L^2_k}.$$

For the left hand side of (22), we have:

$$\begin{aligned}
\nabla_{\mathcal{Q}}\mu(\eta) &= \nabla_{\mathcal{Q}}\mu(\eta_0 + G(\eta_0) + \eta') \\
&= \nabla_{\mathcal{Q}}\mu(\eta_0 + G(\eta_0)) + \int_0^1 \delta_{\eta'} \nabla_{\mathcal{Q}}\mu(\eta_0 + G(\eta_0) + s\eta')ds \\
&= \nabla F(\eta_0) + \delta_{\eta'}\nabla_{\mathcal{Q}}\mu(0) + \int_0^1 (\delta_{\eta'}\nabla_{\mathcal{Q}}\mu(\eta_0 + G(\eta_0) + s\eta') - \delta_{\eta'}\nabla_{\mathcal{Q}}\mu(0))ds
\end{aligned}$$

The first two terms are L^2 orthogonal to each other. For the second term we have

$$||\delta_{\eta'}\nabla_{\mathcal{Q}}\mu(0)||^2_{L^2} = ||H_0(\eta')||^2_{L^2} \geq C \cdot ||\eta'||^2_{L^2_2}.$$

For the last term, we have

$$||\delta_{\eta'}\nabla_{\mathcal{Q}}\mu(\eta_0 + G(\eta_0) + s\eta') - \delta_{\eta'}\nabla_{\mathcal{Q}}\mu(0)||_{L^2} \leq C \cdot ||\eta||_{L^2_k} ||\eta'||_{L^2_2} \leq C \cdot \epsilon \cdot ||\eta'||_{L^2_2}.$$

Therefore, we have

$$||\nabla_{\mathcal{Q}}\mu(\eta)||^2_{L^2} \geq |\nabla F(\eta_0)|^2_{L^2} + C \cdot ||\eta'||^2_{L^2_2}. \tag{24}$$

For the right hand side of (22), we have

$$\begin{aligned}
\mu(\eta) &= \mu(\eta_0 + G(\eta_0) + \eta') \\
&= \mu(\eta_0 + G(\eta_0)) + \int_0^1 \nabla_{\mathcal{Q}}\mu(\eta_0 + G(\eta_0) + s\eta')\eta' ds \\
&= F(\eta_0) + \nabla F(\eta_0)\eta' + \int_0^1 \int_0^1 \delta_{\eta'}\nabla_{\mathcal{Q}}\mu(\eta_0 + G(\eta_0) + st\eta')\eta' dtds \\
&= F(\eta_0) + H_0(\eta')\eta' + \int_0^1 \int_0^1 (\delta_{\eta'}\nabla_{\mathcal{Q}}\mu(\eta_0 + G(\eta_0) + st\eta') - \delta_{\eta'}\nabla_{\mathcal{Q}}\mu(0))\eta' dtds
\end{aligned}$$

So

$$|\mu(\eta) - \mu(0)| \leq |F(\eta_0) - F(0)|_{L^2} + C \cdot ||\eta'||^2_{L^2_2}. \tag{25}$$

Now we apply the usual Lojasiewicz inequality to F, and obtain that

$$|\nabla F(\eta_0)|_{L^2} \geq C \cdot |F(\eta_0) - F(0)|^{\alpha},$$

for some $\alpha \in [\frac{1}{2}, 1)$. Together we have proved (22).

□

4 Proof of the main theorem

Now we shall prove theorem 1.1. Suppose (M, ω, J, g) is a Kähler-Einstein metric of Einstein constant one. Denote by $\mathcal{N}_\delta^{k,\gamma}$ the space of all Kähler structures (ω', J', g') with $\omega' \in 2\pi \cdot c_1(M; J')$ and

$$||g - g'||_{C^{k,\gamma}} + ||J - J'||_{C^{k,\gamma}} \leq \delta.$$

Then there exists a small δ, such that any $(\omega', J', g') \in \mathcal{N}_\delta^{k,\gamma}$ satisfies $c_1(M; J') = c_1(M; J)$, and such that the inequality (12) is satisfied for any Kähler metric $(\omega', J', g') \in \mathcal{N}_\delta^{k,\gamma}$, by lemma 3.1. By definition and lemma 3.3,

$$\mu(g') \leq \mathcal{W}_{g'}(\log \mathrm{Vol}_{g'}(M)) = \frac{1}{2} \int_M R(g')\, \mathrm{dVol}_{g'} = \frac{n}{2} + \log \mathrm{Vol}_{g'}(M),$$

and the equality holds if and only if $Ric(g') = g'$. In particular, we have

$$\mu(g') \leq \mu(g).$$

Since the Kähler-Ricci flow preserves the anti-canonical Kähler condition, we are able to make use of lemma 3.2, and obtain constants $0 < \delta_1 < \delta_2 < \delta$ such that the modified Ricci flow starting from any $g' \in \mathcal{N}_{\delta_1}^{k,\gamma}$ converges in $C^{k,\gamma}$ to a Ricci soliton $g_\infty \in \mathcal{N}_{\delta_2}^{k,\gamma}$ polynomially fast with $\mu(g_\infty) = \mu(g)$. This latter condition implies that g_∞ is indeed Einstein. The complex structure $J(t)$ under the modified Ricci flow evolves as

$$\frac{d}{dt} J(t) = J(t) \mathcal{D}_t f(t),$$

where $\mathcal{D}$ is the Lichnerowicz Laplacian in Kähler geometry. Since

$$\nabla \mu((g(t))) = Ric(g(t)) - g(t) + \mathrm{Hess}_t\, f(t) = [Ric(g(t)) - g(t) + \nabla_t \bar{\nabla}_t f(t)] + \mathcal{D}_t f(t)$$

is a point-wise orthogonal decomposition, we see that

$$||\dot{J}(t)||_{L^2(t)} \leq ||\dot{g}(t)||_{L^2(t)},$$

thus by following the argument in the proof of 3.2, we see that $J(t)$ also converges in $C^{k,\gamma}$ to a limit J_∞ polynomially fast. Moreover, J_∞ and g_∞ are compatible, so the Kähler structures $(\omega(t), J(t), g(t))$ converges in $C^{k,\gamma}$ to the Kähler-Einstein structure $(\omega_\infty, J_\infty, g_\infty)$ in $\mathcal{N}_{\delta_2}^{k,\gamma}$. By lemma 2.3, we conclude the main theorem. The last part follows from [CS]. □

Remark 4.1. For a Kähler-Ricci soliton, since we do not know whether it is always a maximizer of μ among nearby Kähler metrics in a fixed real cohomology class, we can not conclude stability of Kähler-Ricci flow in this case. But similar arguments using lemma 3.2 can show that if the Kähler-Ricci flow converges by sequence to a Kähler-Ricci soliton in the sense of Cheeger-Gromov, then it converges uniformly and polynomially fast in the sense of theorem 1.1. The uniqueness of the limit soliton of a Ricci flow has been proved by N. Sesum([Se2]) under the assumption of integrability.

References

[AL] Claudio Arezzo, Gabriele La Nave. *Complexified Kähler-Ricci flow and families of Kähler-Ricci flow.* Preprint.

[Cao] Huaidong Cao. *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds.* Invent.Math.81(1985), no.2, 359-372.

[CLW] Xiuxiong Chen, Haozhao Li, Bing Wang. *Kähler-Ricci flow with small initial energy.* Geom. Funct. Anal. 18 (2009), no. 5, 1525–1563.

[CS] Xiuxiong Chen, Song Sun. *Calabi flow, geodesic rays, and uniqueness of constant scalar curvature Kähler metrics.* Preprint.

[CT] Xiuxiong Chen, Gang Tian, *Ricci flow on Kähler-Einstein manifolds*, Duke Math. J. 131 (2006), no. 1, 17–73.

[DWW] Xianzhe Dai, Xiaodong Wang, Guofang Wei. *On the variational stability of Kähler-Einstein metrics*, Comm. Anal. Geom., 15 (2007), 669-693.

[Eb] David Ebin. *On the space of Riemannian metrics.* Bull. Amer. Math. Soc. 74 1968 1001–1003.

[Ha] Richard Hamilton. *Three-manifolds with positive Ricci curvature*, J.Differential Geometry. 17(1982), no.2, 255-306.

[KL] Bruce Kleiner, John Lott. *Notes on Perelman's papers*, Geometry and Topology 12(2008), 2587-2855.

[Pe] Grisha Perelman. *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159.

[Ra] Oscar Rothaus. *Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators.* J. Funct. Anal. 42 (1981), no. 1, 110–120.

[Se1] Natasa Sesum. *Linear and dynamical stability of Ricci-flat metrics.* Duke Math. J. 133 (2006), no. 1, 1–26.

[Se2] Natasa Sesum. *Convergence of the Ricci flow toward a soliton.* Comm. Anal. Geom. 14 (2006), no. 2, 283–343.

[Si] Leon Simon. *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems.* Ann. of Math. (2) 118 (1983), no. 3, 525–571.

[TZ1] Gang Tian, Xiaohua Zhu. *Convergence of Kähler-Ricci flow.* J. Amer. Math. Soc. 20 (2007), no. 3, 675–699.

[TZ2] Gang Tian, Xiaohua Zhu. *Perelman's $\mathcal{W}$-functional and stability of Kähler-Ricci flow*, arxiv: math/0801.3504.

[TZ3] Gang Tian, Xiaohua Zhu. *Stability of Kähler-Ricci flow on a Fano manifold II*, preprint.

Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706, U.S.A
Email: ssun@math.wisc.edu; ywang@math.wisc.edu